Elastodynamics of self-gravitating matter: Nonradial vibrations of a star modeled by a heavy spherical mass of an elastic solid

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(Received 16 March 1995)

The continuum dynamics of self-gravitating elastic substance is modeled by the closed system of elastodynamic equations and Poisson's equation of the Newtonian gravity. Instead of the Lamé's equation, which describes small-amplitude vibrations of an isotropic elastic solid, the equations of the elastodynamics are introduced as a natural extension of the hydrodynamic equations: the continuity equation for the bulk density and Euler's equation for the velocity field are supplemented by the equation for the tensor of elastic stresses. The emphasis is placed on the study of nonradial spheroidal and torsional gravitation-elastic vibrations of a star modeled by a heavy spherical mass of a perfectly elastic substance. It is found that eigenfrequencies of spheroidal vibrations are given by $\omega_s^2 = \omega_G^2[2(3L+1)(L-1)/(2L+1)]$; the torsional gravitation-elastic modes are found to be $\omega_t^2 = \omega_G^2(L-1)$, where $\omega_G^2 = 4\pi G\rho_0/3$ is the basic frequency for the star with uniform equilibrium density ρ_0 and where G denotes the gravitational constant. To reveal similarities and differences between the seismology of stars with elastodynamic and fluid-dynamic properties of medium, the vibrational dynamics of a self-gravitating elastic globe is considered in juxtaposition with Kelvin's theory for the small-amplitude oscillations of a heavy spherical drop of an incompressible inviscid liquid.

PACS number(s): 03.40.Dz, 04.40.−b, 43.20.+g, 62.30.+d

I. INTRODUCTION

Studies of nonradial vibrations of variable stars have often relied on the concept of a liquid stellar substance governed by the equations of hydrodynamics and Newtonian gravity [1,2]. The stellar matter is considered to be in the liquid aggregate state mainly in stars from the main sequence. The method underlying fluid dynamical calculation of nonradial eigenmodes of liquid stars is the Kelvin theory of spheroidal vibrations of a homogeneous spherical heavy mass of an incompressible inviscid liquid [3,4]. In the meantime, the vibrational dynamics of stars whose continuum possesses the properties of an elastic solid is less studied. In Ref. [5] arguments have been given that the white dwarfs represent a class of compact objects [6] whose seismology displays elasticlike behavior of highly compressed stellar matter. In Ref. [7], it is emphasized that the variability of white dwarfs is caused primarily by their nonradial pulsations.

The eigenmodes of an elastic self-gravitating object may be investigated on the basis of Lamé's equation, which describes the small-amplitude vibrations of an isotropic elastic solid. This approach is widely used in the terristial seismology when analyzing the surface Rayleigh's waves. The Lamé's equation is particularly efficient in the study of radial eigenmodes that are the result of excitation of standing spherical waves. These modes are specified by the nodal structure of the displacement field. In present paper, a somewhat different mathematical treatment of the dynamics of self-gravitating elastic substance is considered which has been found [8] to be most effective in the search for eigenmodes of the nonradial, long wavelength vibrations of an elastic substance. The characteristic feature of nonradial vibrations is that the displacement field of fluctuating matter not contain nodal points in the spherical volume. The governing equations of the elastodynamics of self-gravitating matter are introduced in Sec. II. These equations have some features in common with the equations of hydrodynamics and are known as equations of the 13 moment's approximation of the continuum theory [9]. The emphasis is placed on the study of the nonradial pulsations of a star modeled by a homogeneous self-gravitating spherical mass of an incompressible continuous medium with an isotropic distribution of stresses in the equilibrium state. In Sec. III, by making use of the Rayleigh variation principle the Hamiltonian is derived for nonradial gravitation-elastic vibrations of a heavy elastic sphere. The eigenfrequencies of spheroidal pulsations are studied in Sec. IV. The problem of spheroidal oscillations has briefly been studied in Ref. [10] within the framework of the Cowling approximation. In this paper, we present a detailed and highly extended analysis of spheroidal gravitation-elastic modes. It should be stressed that the model of a homogeneous spherical distribution of incompressible substance cannot even approximately be considered as an adequate model of the structure of any known class of stars. The principal attractive feature of this model is that it allows one to carry out all calculations analytically and to elucidate the major dynamical differences in the behavior of self-gravitating continuum controlled by hydrodynamic equations of an inviscid liquid and by elastodynamic equations of a perfectly elastic solid. For the purpose of comparison of the hydrody-

1063-651X/96/53(2)/1917(6)/$06.00

namical and elastodynamical models of a heavy continuous medium, in Sec. IV A a brief outline is given of Kelvin's model for nonradial spheroidal vibrations of a heavy drop of an inviscid incompressible liquid. In Sec. V the eigenmodes of torsional gravitation-elastic vibrations are derived. Section VI contains a short summary of the analysis performed.

II. ELASTODYNAMIC EQUATIONS FOR SELF-GRAVITATING CONTINUUM

The mathematical treatment of the continuum dynamics of matter with properties of an elastic solid may be based on the following equations:

$$\frac{d\rho}{dt} + \rho\frac{\partial V_i}{\partial x_i} = 0, \tag{2.1}$$

$$\rho\frac{dV_i}{dt} + \frac{\partial P_{ik}}{\partial x_k} + \rho\frac{\partial U}{\partial x_i} = 0, \tag{2.2}$$

$$\frac{dP_{ij}}{dt} + P_{ik}\frac{\partial V_j}{\partial x_k} + P_{jk}\frac{\partial V_i}{\partial x_k} + P_{ij}\frac{\partial V_k}{\partial x_k} = 0, \tag{2.3}$$

where $d/dt = \partial/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla}$ is the total (convective) derivative, ρ stands for the mass density, V_i is the field of velocity, P_{ij} is the tensor of elastic stresses, and U is the gravitational potential. Equation (2.1) is the continuity equation, Eq. (2.2) describes the flow pattern, and Eq. (2.3) controls the dynamics of internal stresses. The incorporation of this latter equation into the dynamical description of matter actually implies the identification of its behavior with that of ideal elastic continuum. Indeed, the nondiagonal structure of the elastic stress tensor provides the possibility that an external perturbation may produce an anisotropic distortion in the distribution of internal stresses. As shown in Ref. [8] Eqs. (2.1)–(2.3) can be reduced to Lamé's equation describing the vibrations of a perfectly elastic solid, the characteristic feature of which is the ability to support both longitudinal and transverse undamped vibrations. In an inviscid liquid, the perturbation propagates in the form of solely longitudinal vibrations, that is, without spoiling the isotropy of internal stresses. This is the main reason why the continuum model based on the above equations is considered to be the model of an elastic medium. One advantage of these equations is that they have proven to be effective in obtaining the eigenmodes of low-frequency (long wavelength, essentially nonradial) vibrations of a perfectly elastic globe, whereas the approach based on Lamé's equation does not permit a unique solution of this problem. In the continuum theory Eqs. (2.1)–(2.3) are introduced as extensions of the fluid-dynamic equations in the sense that two basic equations of hydrodynamics — the continuity equation (2.1) for the bulk density and the Euler equation (2.2) for the velocity field — are supplemented by the equation for the tensor of elastic stresses (2.3) [9]. In view of this Eqs. (2.1)–(2.3) below are referred to as the equations of elastodynamics.

In this paper we consider the dynamics of elastic matter in the presence of Newtonian gravitation, that is, assuming that equilibrium and large scale motions of matter are dominated by forces of self-gravity. This means that the potential U obeys Poisson's equation

$$\Delta U = 4\pi G\rho\ , \tag{2.4}$$

where G is the gravitation constant. As a result we arrive at the closed system of Eqs. (2.1)–(2.4) governing the dynamics of a self-gravitating elastic continuum.

III. HAMILTONIAN OF NONRADIAL GRAVITATION-ELASTIC VIBRATIONS OF A HEAVY SPHERICAL MASS OF AN ISOTROPIC ELASTIC SOLID

The equilibrium distribution of gravity inside a homogeneous star modeled by a heavy spherical mass of an isotropic matter is given by the well known solution of Eq. (2.4):

$$U_0^{in} = \frac{2\pi}{3}\, G\,\rho_0(r^2 - 3R^2), \qquad r\leq R, \tag{3.1}$$

$$U_0^{ex} = -\frac{4\pi R^3}{3r}\, G\,\rho_0, \qquad r > R, \tag{3.2}$$

where R is the radius of the star and index "0" stands for the characteristics of the star equilibrium. Having presumed that the equilibrium distribution of internal stresses is isotropic, we can write

$$P_{ij}^0(r) = P_0(r)\delta_{ij}. \tag{3.3}$$

The spatial distribution of static stresses in the star interior can be determined from the equation of equilibrium with the boundary condition corresponding to the free surface

$$\nabla P_0(r) = -\rho_0\nabla U_0^{in}(r), \qquad P_0(r)\Big|_{r=R} = 0, \tag{3.4}$$

whose solution is well known:

$$P_0 = \frac{2\pi}{3}\, G\,\rho_0^2(R^2 - r^2). \tag{3.5}$$

The above-defined static characteristics of a star with uniform density and isotropic distribution of equilibrium stresses are the same for two types of stellar matter: inviscid liquid and isotropic elastic solid. The main purpose of our further analysis is to reveal the dynamical difference between self-gravitating spherical masses in which the small-amplitude vibrations of matter are controlled by hydrodynamic equations (2.1)–(2.2) and those in which continuum is governed by equations of elastodynamics (2.1)–(2.3).

To compute the fundamental frequencies of normal vibrations, we take advantage of the Rayleigh variational principle. In what follows we assume that perturbation does not lead to the fluctuations in the density, i.e., $\delta\rho = 0$, and the flow of mass in an equilibrium state is absent $V_0 = 0$. Making use of the standard procedure of linearization, Eqs. (2.1)–(2.4) are reduced to

$$\frac{\partial \delta V_i}{\partial x_i} = 0, \quad (3.6)$$

$$\rho_0 \frac{\partial \delta V_i}{\partial t} + \frac{\partial \delta P_{ij}}{\partial x_j} + \rho_0 \frac{\partial \delta U}{\partial x_i} = 0, \quad (3.7)$$

$$\frac{\partial \delta P_{ij}}{\partial t} + P_0 \left(\frac{\partial \delta V_i}{\partial x_j} + \frac{\partial \delta V_j}{\partial x_i} \right) + \delta_{ij} \left(\delta V_k \frac{\partial P_0}{\partial x_k} \right) = 0, \quad (3.8)$$

$$\Delta \delta U = 0. \quad (3.9)$$

Equations (3.6) and (3.7) have some features in common with the equations for small-amplitude fluctuations in an incompressible inviscid liquid [see below, Eqs. (4.18) and (4.19)].

Taking the scalar product of (3.7) with δV_i and integrating over the star volume, we arrive at the equation of energy balance:

$$\frac{\partial}{\partial t} \int_V \frac{1}{2} \rho_0 \delta V^2 d\tau - \int_V \delta P_{ij} \frac{\partial \delta V_i}{\partial x_j} d\tau + \oint_S [\rho_0 \, \delta U \delta V_i + \delta P_{ij} \, \delta V_j] \, d\sigma_i = 0, \quad (3.10)$$

which determines the conservation of energy in the process of oscillations. It is convenient to represent the velocity field of the perturbed flow δV_i and the fluctuation in the potential of self-gravity δU in the separable form

$$\delta V_i(\mathbf{r}, t) = \xi_i^L(\mathbf{r}) \dot{\alpha}_L(t), \qquad \delta U(\mathbf{r}, t) = \phi^L(\mathbf{r}) \alpha_L(t), \quad (3.11)$$

where L is the multipole order of an oscillation. The normal coordinate $\alpha_L(t)$ specifies the time dependence of fluctuating variables and $\boldsymbol{\xi}^L(\mathbf{r})$ represents the field of instantaneous displacements. Substituting (3.11) into (3.8), one finds that fluctuations in stresses are determined by the tensor

$$\delta P_{ij}(\mathbf{r}, t) = - \left[P_0(\mathbf{r}) \left(\frac{\partial \xi_i^L(\mathbf{r})}{\partial x_j} + \frac{\partial \xi_j^L(\mathbf{r})}{\partial x_i} \right) + \delta_{ij} \left(\xi_k^L(\mathbf{r}) \frac{\partial P_0(\mathbf{r})}{\partial x_k} \right) \right] \alpha_L(t). \quad (3.12)$$

In view of the above separation of the spatial and time dependence of fluctuating variables, the substitution of (3.11) and (3.12) in the equation of energy balance (3.10) allows one to reduce the latter equation to the Hamiltonian of normal vibrations,

$$H = \frac{\mathcal{M}_L \dot{\alpha}_L^2}{2} + \frac{\mathcal{K}_L \alpha_L^2}{2}, \quad (3.13)$$

which is an integral of motion. The parameters of inertia $\mathcal{M}_L$ and stiffness $\mathcal{K}_L$ are given by

$$\mathcal{M}_L = \int_V \rho_0 \xi_i{}^L \xi_i{}^L \, d\tau, \quad (3.14)$$

$$\begin{aligned} \mathcal{K}_L &= \int_V P_0 \left(\frac{\partial \xi_i{}^L}{\partial x_j} + \frac{\partial \xi_j^L}{\partial x_i} \right) \frac{\partial \xi_j^L}{\partial x_i} d\tau + \oint_S \left(\rho_0 \phi^L - \xi_j^L \frac{\partial P_0}{\partial x_j} \right) \xi_i^L \, d\sigma_i \quad (3.15) \\ &= \frac{1}{2} \int_V P_0 \left(\frac{\partial \xi_i{}^L}{\partial x_j} + \frac{\partial \xi_j^L}{\partial x_i} \right)^2 d\tau + \oint_S \left(\rho_0 \phi^L - \xi_j^L \frac{\partial P_0}{\partial x_j} \right) \xi_i^L \, d\sigma_i . \end{aligned}$$

The first term in Eq. (3.15) explicitly displays the contribution of energy of elastic anisotropic deformations to the total potential energy of nonradial oscillations. Equation (3.15) for the stiffness $\mathcal{K}_L$ has been obtained under the condition $P_0(R) = 0$. Thus, to determine the fundamental frequencies $\omega_L^2 = \mathcal{K}_L / \mathcal{M}_L$ of nonradial gravitation-elastic oscillations of a star, it is necessary to calculate the spatial distributions of a fluctuating field of the velocity δV_i (more exactly, of the field of displacements ξ_i^L) and of fluctuations in the gravity potential δU (precisely, the function ϕ^L).

IV. SPHEROIDAL GRAVITATION-ELASTIC MODES

Spheroidal oscillations are defined as those under which an arbitrary spherical surface in the volume of a star transforms into a harmonic spheroid whose surface is described by the equation

$$r'(t) = r[1 + \alpha_L(t) P_L(\cos\theta)], \quad (4.1)$$

where r is the radius of the unperturbed spherical surface; $P_L(\cos\theta)$ is the Legendre polynomial of order L (hereafter all the calculations are performed in the system with a fixed polar axis). Spheroidal oscillations are accompanied by an irrotational vector field of instantaneous displacements $\boldsymbol{\xi}^L(\mathbf{r}) = \text{grad}\,\psi^L$. Inserting this field into the equation of incompressibility (3.6), one has

$$\Delta \psi^L = 0, \qquad \psi^L = A_L r^L P_L(\cos\theta). \quad (4.2)$$

Equation (4.2) can be considered as a long-wavelength limit of the Helmholtz equation for a spherical standing wave $\Delta\psi + k^2 \psi = 0$. Indeed, in the limit of long wavelengths $\lambda \to \infty$, the wave number $k = 2\pi/\lambda \to 0$, and the Helmholtz equation turns into the vector equation of Laplace (4.2). An arbitrary constant A_L is fixed by the condition

$$\delta V_r(r') = \dot{r}'|_{r'=R', (r=R)}, \quad (4.3)$$

which expresses the compatibility of fluctuations of the radial component of the velocity field with the speed of distortions of the surface given by Eq. (4.1). The displacement field for spheroidal oscillations is finally written as follows:

$$\xi_i^L = \frac{1}{L R^{L-2}} \frac{\partial}{\partial x_i} r^L P_L(\cos\theta). \quad (4.4)$$

The parameter of inertia $\mathcal{M}_L$, Eq. (3.14), computed with the field (4.4) is given by

$$\mathcal{M}_L = \frac{4\pi\rho_0 R^5}{L(2L+1)}. \quad (4.5)$$

The general solutions of the Laplace equation (3.9) for the field of the self-gravity δU are

$$\delta U^{in} = B_L\, r^L\, P_L(\cos\theta)\alpha_L, \qquad r \leq R, \quad (4.6)$$

$$\delta U^{ex} = C_L\, r^{-(L+1)}\, P_L(\cos\theta)\alpha_L, \qquad r > R. \quad (4.7)$$

The arbitrary constants B_L and C_L are fixed by the standard boundary conditions

$$U_0^{in}(r') + \delta U^{in}(r') = U_0^{ex}(r') + \delta U^{ex}(r')|_{r'=R',(r=R)}, \quad (4.8)$$

$$\frac{\partial U_0^{in}(r')}{\partial r'} + \frac{\partial \delta U^{in}(r')}{\partial r'} = \frac{\partial U_0^{ex}(r')}{\partial r'} + \frac{\partial \delta U^{ex}(r')}{\partial r'}\bigg|_{r'=R',(r=R)}. \quad (4.9)$$

Inserting (4.6) and (4.7) into (4.8) and (4.9) and retaining the terms linear in α_L, one obtains

$$\delta U^{in} = -\frac{4\pi}{R^{L-2}}\frac{G\,\rho_0}{(2L+1)}r^L P_L(\cos\theta)\alpha_L \quad (4.10)$$

and

$$\delta U^{ex} = -\frac{4\pi\, G\rho_0 R^{L+3}}{(2L+1)}r^{-(L+1)}P_L(\cos\theta)\alpha_L. \quad (4.11)$$

It follows from (4.11) that function ϕ^L on the star surface is given by

$$\phi^L = -\frac{4\pi\, G\rho_0 R^2}{(2L+1)}P_L(\cos\theta). \quad (4.12)$$

A calculation of the volume part of the stiffness parameter (3.15) yields

$$\frac{1}{2}\int_V P_0\left(\frac{\partial\xi_i^L}{\partial x_j} + \frac{\partial\xi_j^L}{\partial x_i^L}\right)^2 d\tau = \frac{32}{3}\pi^2\, G\,\rho_0^2\, R^5\frac{(L-1)}{L(2L+1)}. \quad (4.13)$$

For the surface integral in Eq. (3.15) one obtains

$$\oint_S\left(\rho_0\phi^L - \xi_j^L\frac{\partial P_0}{\partial x_j}\right)\xi_i^L\, d\sigma_i = \frac{32}{3}\pi^2\, G\,\rho_0^2\, R^5\frac{(L-1)}{(2L+1)^2}. \quad (4.14)$$

Finally, the stiffness coefficient of spheroidal gravitation-elastic pulsations is expressed as follows:

$$\mathcal{K}_L = \frac{32}{3}\pi^2\, G\,\rho_0^2\, R^5\frac{(3L+1)(L-1)}{L(2L+1)^2}. \quad (4.15)$$

From (4.15) it follows that monopole and dipole modes cannot exist. The excitation of monopole (purely radial at $L = 0$) pulsations is impossible due to assumption that the density is not affected by perturbation. When the dipole field of velocity is disturbed, only motion of the center-of-mass of a star can be expected, without change of its internal state, since the stiffness coefficient at $L = 1$ vanishes.

The eigenmodes of nonradial spheroidal gravitation-elastic pulsations are uniquely determined by the frequencies

$$\omega_s^2 = \omega_G^2\frac{2(3L+1)(L-1)}{(2L+1)}, \quad (4.16)$$

with

$$\omega_G^2 = \frac{4}{3}\pi\, G\,\rho_0. \quad (4.17)$$

Formula (4.16) is one of the basic consequences of the model in question. It shows that the eigenfrequency of spheroidal gravitation-elastic modes does not depend on the star radius but only on the density of homogeneous incompressible stellar continuum.

Comparison with the Kelvin modes

To ascertain the difference in the dynamics of a homogeneous elastic and liquid stars we present a short derivation of the Kelvin eigenmodes by using the variational principle expounded above.

The hydrodynamics of inviscid liquid assumes that disturbance of the equilibrium state does not destroy the isotropy in the distribution of static internal stains: $P_{ij} = (P_0 + \delta P)\delta_{ij}$. This is the main feature distinguishing an inviscid liquid from a perfectly elastic solid. Propagation of disturbance in the elastic substance, as was shown above, is accompanied by the spoiling the isotropy of the equilibrium distribution of stresses: $P_{ij} = P_0\delta_{ij} + \delta P_{ij}$, with δP_{ij} given by Eq. (3.12).

In the linear approximation, the evolution of the disturbances in an inviscid self-gravitating incompressible liquid is described by the equations

$$\frac{\partial\delta V_i}{\partial x_i} = 0, \quad (4.18)$$

$$\rho_0\frac{\partial\delta V_i}{\partial t} + \frac{\partial\delta P}{\partial x_i} + \rho_0\frac{\partial\delta U}{\partial x_i} = 0, \quad (4.19)$$

$$\Delta\delta U = 0. \quad (4.20)$$

Taking the scalar product of the linearized Euler equation (4.19) with V_i and integrating over the volume of star, we again arrive at the equation of energy balance:

$$\frac{\partial}{\partial t}\int_V \rho_0\frac{\delta V^2}{2}d\tau + \oint_S(\delta P + \rho_0\delta U)\delta V_i d\sigma_i = 0. \quad (4.21)$$

Fluctuations in the velocity δV_i and in the potential of self-gravity δU are calculated in the same way as in the previous section. The only unknown variable is the variation of pressure δP. Acting by the divergence operator on Eq. (4.19) and using Eqs. (4.18) and (4.20), one finds that δP obeys Laplace's equation together with the condition for vanishing pressure on the pulsating surface:

$$\Delta\delta P = 0, \qquad P_0(r') + \delta P(r') = 0|_{r'=R',(r=R)}. \quad (4.22)$$

The equilibrium pressure P_0 is defined by the expression (3.5). The solution to Eq. (4.22) is of the form

$$\delta P = p^L(\mathbf{r})\,\alpha_L(t), \qquad p^L(\mathbf{r}) = \frac{4\pi}{3R^{L-2}}\,G\,\rho_0^2\,r^L P_L. \quad (4.23)$$

Substituting (3.11), (4.12), and (4.23) into (4.21) we arrive at the standard equation of normal oscillations

$$\mathcal{M}_L\ddot{\alpha}_L^2 + \mathcal{K}_L\alpha_L^2 = 0. \quad (4.24)$$

The mass parameter $\mathcal{M}_L$ is defined by the expression (4.5) and the stiffness parameter equals

$$\mathcal{K}_L = \oint_S \left(p^L + \rho_0\phi^L\right)\xi_i^L d\sigma_i = \frac{32}{3}\pi^2\,G\,\rho_0^2\,R^5\frac{(L-1)}{(2L+1)^2}. \quad (4.25)$$

As a result we arrive at Kelvin's formula for eigenfrequencies of nonradial vibrations of a heavy spherical mass of an inviscid incompressible liquid:

$$\omega_L^K = \omega_G^2\frac{2L(L-1)}{2L+1}. \quad (4.26)$$

It is remarkable that spheroidal modes of a self-gravitating elastic sphere (4.16) have many features in common with Kelvin modes (4.26). In both cases the eigenmodes of nonradial spheroidal pulsations solely depend on the equilibrium density and the lowest mode is the quadrupole one. However, the ratio between these frequencies obeys the inequality

$$\frac{\omega_s^2}{(\omega_L^K)^2} = \frac{(3L+1)}{L} > 1, \qquad L \geq 2, \quad (4.27)$$

from which it follows that $\omega_s \to \sqrt{3}\omega_L^K$ when $L \to \infty$. So, for the same L and ρ_0, the frequencies of spheroidal oscillations of a heavy elastic sphere are always higher than those for a self-gravitating spherical mass of inviscid liquid.

V. TORSIONAL GRAVITATION-ELASTIC MODES

The elasticity of stellar medium allows one to consider shear torsional gravitation-elastic oscillations of a homogeneous star. This kind of vibration cannot be excited in the liquid star [4], because shear oscillations are due to the appearance of anisotropic distortions in the distribution of internal stresses. The torsional long-wavelength oscillations are described by the toroidal field of velocity [8]:

$$\delta\mathbf{V} = A_L\,\mathrm{rot}\,\mathbf{r}\,r^L P_L(\cos\theta)\dot{\alpha}_L(t) = [\mathbf{r}\times\mathbf{\Omega}(\mathbf{r},t)], \quad (5.1)$$

where

$$\mathbf{\Omega}(\mathbf{r},t) = -A_L\,\mathrm{grad}\,r^L P_L(\cos\theta)\dot{\alpha}_L(t) \quad (5.2)$$

is the frequency of local (nonrigid) rotational oscillations. In this case the normal coordinate $\alpha_L(t)$ represents the azimuthal angle of torsion of the flow around the polar axis. The arbitrary constant A_L may be fixed by the following boundary condition:

$$\delta\mathbf{V} = [\mathbf{r}\times\mathbf{\Omega}_0]\,|_{r=R}, \quad (5.3)$$

where $\mathbf{\Omega}_0 = \mathrm{grad}\,P_L(\cos\theta)\dot{\alpha}_L(t)$, which yields $A_L = R^{-L+1}$. The field of instantaneous torsional displacements is given by

$$\boldsymbol{\xi} = \frac{1}{R^{L-1}}\,\mathrm{rot}\,\mathbf{r}\,r^L P_L(\cos\theta). \quad (5.4)$$

Inserting (5.4) into (3.14), we obtain

$$\mathcal{M}_L = 4\pi\rho_0 R^5\frac{L(L+1)}{(2L+1)(2L+3)}. \quad (5.5)$$

This is the moment of inertia of rotational oscillations. When $L = 1$, the vorticity field $\mathbf{\Omega}$ becomes uniform and the mass parameter $\mathcal{M}_1$ coincides with the moment of inertia of a hard sphere $J = \mathcal{M}_1 = (2/5)MR^2$. The stiffness of torsional gravitation-elastic vibrations equals

$$\mathcal{K}_L = \frac{16\pi^2}{3}\,G\,\rho_0^2\,R^5\frac{L(L-1)(L+1)}{(2L+1)(2L+3)}. \quad (5.6)$$

It worth noticing that the surface integral in the general expression for the stiffness (3.15) vanishes. Consequently, surface fluctuations in the gravity potential do not contribute to the restoring force of shear oscillations. The latter means that the torsional vibrations of a star are of the volume origin. From (5.6) it follows that the dipole torsional mode, as in the case of spheroidal oscillations, is not an eigenmode of oscillator Hamiltonian (3.13). When the toroidal dipole field of displacements is excited, the restoring force does not arise, and this motion corresponds to rigid rotations of a star without changing the intrinsic state.

The homogeneous model under consideration leads to the following expression for frequencies of torsional gravitation-elastic modes:

$$\omega_t^2 = \omega_G^2(L-1), \quad (5.7)$$

where ω_G is the fundamental frequency defined by (4.17).

VI. SUMMARY

In this paper an elastodynamical model is explored of a self-gravitating continuum based on Eqs. (2.1)–(2.4). The major purpose was to construct equations that are able to reflect the basic property of dynamical response of a heavy spherical mass of perfectly elastic matter, that is, spheroidal (longitudinal) and torsional (transverse) vibrations (as compared to a heavy nonviscous liquid drop, which may execute solely spheroidal vibrations). To reveal the difference between the elastic and liquid behaviors of an isotropic stellar matter, the dynamics of vibrations has been analyzed for those perturbations that do not lead to fluctuations in density. As a representative example of the method, the nonradial spheroidal and torsional pulsations have been studied of a star in the model of homogeneous spherical mass of a self-gravitating matter governed by equations of elastodynamics. The inherent feature of nonradial oscillations is that the bulk density of elastic matter remains unchanged, and in this respect the elastic substance under consideration bears a resemblance to incompressible liquid. The eigenfrequencies of gravitation-elastic modes have been derived in analytic form analogous to that for the Kelvin modes for a heavy spherical mass of an inviscid incompressible liquid. The frequency of spheroidal gravitation-elastic vibrations of a self-gravitating elastic globe, Eq. (4.16), is found to be always higher in absolute value than the Kelvin frequency of nonradial vibrations of a heavy liquid drop, (4.26), at equal densities of stellar continuum. This difference is of a purely dynamical origin: the restoring force in the pulsating spherical mass of a heavy incompressible nonviscous liquid is determined only by the surface fluctuations in the gravity and pressure. The restoring force of spheroidal oscillations of a heavy elastic globe is dominated by two effects of volume and surface origin that act constructively. The first of these is associated with anisotropic distortions of internal stresses caused by self-gravitation in the star volume. The physical content of the surface restoring force is similar to that for a heavy drop of inviscid liquid. The elastodynamical method in question allows one to obtain an analytic expression for the eigenfrequency of torsional (essentially transverse) gravitation-elastic oscillations, Eq. (5.7). These modes are unique to the star with elastic continuum. In the stars whose medium is governed by hydrodynamical equations, excitation of shear modes cannot be expected.

As was mentioned above, nowadays it is believed that some observable features of white dwarfs may be understood if one accepts that a stellar continuum displays elasticlike behavior. Therefore one may hope that the elastodynamical method considered may be useful in the study of seismology of highly condensed stars.

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